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Validation of Reduced-Order Models for Control Systems Design

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The concept of suboptimality is applied to testing validity of reduced-order models in design of feedback schemes for large-scale systems. Aggregation and singular value decomposition, as model reduction techniques, are interpreted in the expansion-contraction framework, which is suitable for evaluation of suboptimality of closed-loop systems resulting from reduced-order designs. The proposed validation procedure is applied to a control design of a large space structure.

I. Introduction

IN system theory and applications, "model reduction" stands for a variety of more or less disparate concepts and techniques which have a common goal: to reduce the size of the mathematical model of a large system in order to simplify the design of control and estimation schemes. Due to the nature of the mathematical models involved in describing the motion of large space structures (LSS), a large number of recent results in model reduction have been obtained in this field, with emphasis on the size and complexity of the controller to be implemented onboard the spacecraft (see, for example, Refs. 1 and 2). Most of the proposed model reduction procedures have a common plan. First, the elements of a mathematical model are separated into two parts according to the significance of their contribution to the desired performance characteristic of the system (stability, optimality, frequency response, and the like). Second, the most significant part of the model is used to design the control scheme for the overall system. Finally, the scheme is implemented and the system performance is tested usually by an extensive simulation analysis. The emphasis on each phase of the model reduction plan varies from concept to concept, but the first phase involving the decomposition of the models has been the most considered area in the research on model reductions.

Almost exclusively, all the model reduction techniques are concerned with the open-loop behavior of the system and are, therefore, subject to the question of whether the reduced-order model would still be a good representation of the overall system after the feedback loop is closed. The answer to this question would most probably be positive if a reduced-order model were indeed a good approximation of the original system. Yet, it is difficult to say how good the approximation is without a comparison of the resulting closed-loop system with the one that would be obtained under ideal conditions when a full controller is designed using the entire model of the system.

In order to resolve the validation problem of model-reduction schemes, we note that there are two distinct simplifications attempted by the methods: to obtain a simple control law which is suitable for implementation and to reduce the computations involved in the control design. Since the off-line computations needed to determine the best control law for dynamic systems of considerable size may well be within the capabilities of today's computers, the dimensionality problem is not as severe as the compatibility problem of the control structure with the design constraints. For this reason, an appropriate approach to the model-reduction problem is to concentrate on construction of suitable control laws and, subsequently, compare the resulting closed-loop system with the optimal off-line model used as reference. This approach to model reduction is in the spirit of suboptimal design schemes developed in the context of large-scale systems[3,4] and the major objective of this paper is to adapt these suboptimality schemes for use in model-reduction problems. This reformulation, in turn, opens up a real possibility for future use of the model-reduction approach in the design of modern decentralized control schemes for large space structures composed of interconnected systems.

The organization of the paper is as follows. In the next section, we briefly summarize the concept of suboptimality which provides a basis for the subsequent development. In Sec. III, we outline the inclusion principle, which constitutes an appropriate mathematical framework for perfect model reduction. In that section, we also study the role of inclusion principle in the context of suboptimal control, and show that when perfect model reduction is possible, then the suboptimal control obtained from the reduced-order model is actually optimal for the original overall system. In Sec. IV, we elaborate on various approximate model-reduction schemes. From the point of view of suboptimality, these schemes appear as deviations from perfect model reductions obtained via the inclusion principle. In Sec. V, a model-reduction technique based upon the balanced representations is applied to a model of a large space structure. Finally, in conclusion, we discuss possible extensions of the suboptimal approach to more elaborate problems of LSS.

II. Suboptimal Control

Let us consider a linear system $\mathcal{S}$ described as

$$\mathcal{S}\colon \dot{x} = Ax + Bu, \qquad x(0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input of $\mathcal{S}$, and the matrices A and B are constant and of appropriate

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dimensions. With $\mathcal{S}$ we associate a quadratic performance index

$$J=\int_0^\infty (x^TQx+u^TRu)\,\mathrm{d}t \tag{2}$$

where Q is a symmetric non-negative definite matrix and R a symmetric positive definite matrix, both matrices being constant and of appropriate dimensions. The system $\mathcal{S}$ and the cost functional J form a standard optimization problem $\{\mathcal{S}, J\}$, the solution of which is given as

$$u^\circ=-K^\circ x \tag{3}$$

when the gain matrix K° is

$$K^\circ=R^{-1}B^TP \tag{4}$$

and P is the solution of the Riccati equation

$$A^TP+PA-PBR^{-1}BP+Q=0 \tag{5}$$

Application of the control u° to $\mathcal{S}$ results in the optimal closed-loop system

$$\mathcal{S}^\circ\!: \dot{x}=(A-BK^\circ)x \tag{6}$$

and yields the optimal cost $J^\circ(x_0)=J(x_0,K^\circ)$,

$$J^\circ(x_0)=x_0^TPx_0 \tag{7}$$

Now, suppose instead of the optimal control u°, we apply another control

$$u=-Kx \tag{8}$$

where the gain matrix K is obtained by some rational procedure which is different from the preceding optimal solution of $\{\mathcal{S}, J\}$. Then, the control u of Eq. (8) produces a cost $J^\oplus(x_0)=J(x_0,K)$, which is, in general, larger than the optimal cost $J^\circ(x_0)$. In fact, the cost $J^\oplus(x_0)$ may very well be infinite if the control in Eq. (8) is not stabilizing. The following statement is now made.[4]

Definition 1. The control law of Eq. (8) is said to be suboptimal with degree μ for $\mathcal{S}$ with respect to $\mathcal{S}^\circ$ if there exists a positive number μ such that

$$J^\oplus(x_0)\le\mu^{-1}J^\circ(x_0) \tag{9}$$

for all x_0.

To derive conditions for suboptimality and compute the suboptimality index μ, we note that if the control of Eq. (8) is stabilizing for $\mathcal{S}$, then the suboptimal cost is given by

$$J^\oplus(x_0)=x_0^THx_0 \tag{10}$$

where H is the unique, symmetric, and positive definite solution of the Liapunov equation

$$(A-BK)^TH+H(A-BK)+Q+KRK=0 \tag{11}$$

This leads to the following.[5,6]

Theorem 1. If the control law of Eq. (8) is stabilizing for $\mathcal{S}$, then it is suboptimal for $\mathcal{S}$ with degree

$$\mu=\lambda_M^{-1}(HP^{-1}) \tag{12}$$

where λ_M denotes the largest eigenvalue of the indicated matrix.

An alternative suboptimality criterion, which does not require a test for stability, is provided by the following.[4]

Theorem 2. The control law of Eq. (8) is suboptimal for $\mathcal{S}$ with degree μ if the matrix

$$F(\mu)=(K^\circ-K)^TR(K^\circ-K)-(1-\mu)(Q+K^TRK) \tag{13}$$

is negative semidefinite.

If suboptimality is established by theorem 2, then the suboptimal control law of Eq. (8) is a stabilizing control law for $\mathcal{S}$ provided the pair of matrices $[A-BK,-F^{1/2}(0)]$ is observable (detectable).[5]

We note that to compute the suboptimality index μ using theorem 1 or 2, we have to compute the solution of the problem $\{\mathcal{S}, J\}$. Although we know the optimal control of Eq. (3), we may prefer to apply the suboptimal control of Eq. (8). One important reason might be that the optimal control is difficult to implement in the actual design. For example, we may choose the output feedback

$$u=-Ky \tag{14}$$

where

$$y=Cx \tag{15}$$

is the only available measurement with the matrix C having a full row rank. In this case, a reasonable choice for K would be[7]

$$K=K^\circ C^T(CC^T)^{-1}C \tag{16}$$

which is the projection of K° on the row space of the output matrix C. We do not gain anything computationally, but only satisfy the design constraints on the control structure. Another way to generate the gain matrix K in Eq. (8), which satisfies the information structure constraint, is to replace by zeros those elements of K° that correspond to states that are not available for feedback. Following Ref. 4, both types of control are referred to as *degenerate control*.

Alternatively, the control of Eq. (8) can be generated from the solution of a smaller problem $\{\bar{\mathcal{S}}, \bar{J}\}$ where the system

$$\bar{\mathcal{S}}\!: \dot{\bar{x}}=\bar{A}\bar{x}+\bar{B}u, \qquad \bar{x}(0)=\bar{x}_0 \tag{17}$$

represents a reduced order model of $\mathcal{S}$, that is, $\bar{x}(t)\in\mathcal{R}^{\bar{n}}$ and $\bar{n}\le n$. The cost

$$\bar{J}=\int_0^\infty (\bar{x}^T\bar{Q}\bar{x}+uRu)\,\mathrm{d}t \tag{18}$$

is obtained from the cost J in a manner compatible with the reduction of $\mathcal{S}$ to $\bar{\mathcal{S}}$. Usually, $\bar{\mathcal{S}}$ can be chosen as a projection of $\mathcal{S}$ onto some subspace of the state space[5,6] defined by a singular transformation

$$\bar{x}=Ux \tag{19}$$

Now, if the optimal solution of the problem $\{\bar{\mathcal{S}}, \bar{J}\}$ is obtained as

$$u=-\bar{K}^\circ\bar{x} \tag{20}$$

then the control

$$u=-\bar{K}^\circ Ux \tag{21}$$

can be a reasonable choice for the original system $\mathcal{S}$. Suboptimality of the control of Eq. (21) can be tested using theorem 1 or 2 with $K=\bar{K}^\circ U$.

Another advantage of obtaining the suboptimal control by the solution of a smaller optimization problem $\{\bar{\mathcal{S}}, \bar{J}\}$, is a possibility to compare the suboptimal cost

$J^{\oplus}(x_0) = J(x_0, K^\circ U)$ with the optimal cost $\tilde{J}^\circ(\tilde{x}_0) = \tilde{J}(\tilde{x}_0, \tilde{K}^\circ)$ of pair $\{\tilde{S}, \tilde{J}\}$. Of course, $\tilde{x}_0$ should be chosen as the projection of x_0, that is, $\tilde{x}_0 = Ux_0$. If this comparison can be worked out, then we can avoid solving the original problem $\{S, J\}$, thus providing for both computational and implementational simplifications. In the context of suboptimality,[4,5] taking the optimal small system $\tilde{S}^\circ$ as reference in the definition of suboptimality is the dual of taking the original system S° as the reference system, in the sense that in the latter case the control is considered to be perturbed from the optimal value, while in the former case the system is perturbed by the residuals that were eliminated by the projection.

Finally, we should mention the fact that suboptimality concept has been used in the modal cost reduction scheme[10,11] where the suboptimality index μ has been identified as *model error index* $Q = 1 - \mu^{-1}$ with μ restricted to the interval $[1, +\infty)$. This restricted version of the index μ, which appeared in the early works on suboptimal control design (e.g., Ref. 3), excludes the possibility of a perturbed system to be better than the original optimal system, a possibility present, for example, in the approximate aggregation by decomposition[8] considered in Sec. IV.

III. Inclusion Principle

In using a reduced-order model to build a controller, it is important to know if the behavior of the original system can be predicted by the model. The inclusion principle of Refs. 5 and 6 provide conditions under which the reduced-order model represents perfectly the motion of the original system. In this section, we briefly summarize the principle and discuss its implications.

Let us reconsider the two systems S and $\tilde{S}$, which are described now as

$$S: \dot{x} = Ax + Bu, \qquad x(0) = x_0 \qquad y = Cx \tag{22}$$

and

$$\tilde{S}: \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad \tilde{x}(0) = \tilde{x}_0 \qquad \tilde{y} = \tilde{C}\tilde{x} \tag{23}$$

where the outputs $y(t) \in \mathbb{R}^\ell$ and $\tilde{y}(t) \in \mathbb{R}^\ell$ are added to the previous descriptions of Eqs. (1) and (17). The solutions of Eqs. (22) and (23) are denoted by $x(t, x_0; u)$ and $\tilde{x}(t, \tilde{x}_0; u)$, respectively. We have the following statement.

Definition 2. The system S of Eq. (22) is said to include the system $\tilde{S}$ of Eq. (23) if there exist matrices U and V of order $\tilde{n} \times n$ and $n \times \tilde{n}$ such that $UV = I_{\tilde{n}}$, and

$$\tilde{x}(t, \tilde{x}_0; u) = Ux(t, V\tilde{x}_0; u) \tag{24}$$

and

$$\tilde{y}(t) = y(t) \tag{25}$$

for all t, $\tilde{x}_0$, and $u(t)$.

Definition 2 implies that if S includes $\tilde{S}$, then the system S contains all the necessary information about the system $\tilde{S}$. If S includes $\tilde{S}$, then S is said to be an *expansion* of $\tilde{S}$, and $\tilde{S}$ is a *contraction* of S. In definition 2, $I_{\tilde{n}}$ denotes the $\tilde{n} \times \tilde{n}$ identity matrix. Obviously, an interesting question is: Under what conditions does S include $\tilde{S}$? This is provided by the following.[5]

Theorem 3. S includes $\tilde{S}$ if, and only if, there exist matrices U and V such that $UV = I_{\tilde{n}}$ and

$$UA^i V = \tilde{A}^i, \quad UA^{i-1}B = \tilde{A}^{i-1}\tilde{B}, \quad CA^{i-1}V = \tilde{C}\tilde{A}^{i-1},$$

$$CA^{i-1}B = \tilde{C}\tilde{A}^{i-1}\tilde{B} \qquad i = 1, 2, \ldots, n \tag{26}$$

Let us consider some special cases. First, suppose there exists an $\tilde{n} \times n$ matrix U with full row rank such that

$$UA = \tilde{A}U, \qquad UB = \tilde{B}, \qquad C = \tilde{C}U \tag{27}$$

Then, it is easy to show that the conditions in Eq. (26) are satisfied for any right inverse V of U, that is, for any V such that $UV = I_{\tilde{n}}$. Therefore, if the conditions of Eq. (27) are satisfied for some U, then $\tilde{S}$ is a contraction of S and, hence, Eqs. (24) and (25) hold. However, in this special case, there is a stronger relation between the solutions of S and $\tilde{S}$, namely,

$$\tilde{x}(t, Ux_0; u) = Ux(t, x_0; u) \tag{28}$$

for all t, x_0, and $u(t)$. In other words, whatever the initial state x_0, a projection of the solution of S can be recovered from a solution of $\tilde{S}$. In this case, $\tilde{S}$ is an *aggregation* of S in the sense of Aoki.[8]

As another special case, suppose there exists an $n \times \tilde{n}$ matrix V with full column rank such that

$$AV = V\tilde{A}, \qquad B = V\tilde{B}, \qquad CV = \tilde{C} \tag{29}$$

Then, for any left inverse U of V satisfying $UV = I_{\tilde{n}}$, conditions of Eq. (26) hold, and $\tilde{S}$ is a contraction of S. In this case, too, a stronger relation holds,

$$V\tilde{x}(t, \tilde{x}_0;\ u) = x(t, V\tilde{x}_0;\ u) \tag{30}$$

which implies that solutions of S starting in a subspace of the state space of S can be recovered from the solutions of $\tilde{S}$. In this case we say that $\tilde{S}$ is a *restriction* of S.

Let us now broaden the inclusion principle to consider the optimization problems via model reduction. In the optimal control problem $\{S, J\}$, we let

$$Q = C^T C \tag{31}$$

from some matrix C and rewrite J as

$$J = \int_0^\infty (y^T y + u^T R u)\, dt \tag{32}$$

where $y(t)$ may be taken as the output of S described by Eq. (22). Let us assume that the system S of Eq. (22) is aggregable to a system $\tilde{S}$ described by Eq. (23). With $\tilde{S}$ we associate the cost

$$\tilde{J} = \int_0^\infty (\tilde{y}^T \tilde{y} + u^T R u)\, dt \tag{33}$$

We choose a control

$$u = -\tilde{K}\tilde{x} \tag{34}$$

which is not necessarily optimal for $\{\tilde{S}, \tilde{J}\}$, and apply to $\tilde{S}$. Under the assumption that it is stabilizing, the control of Eq. (34) yields the cost given by

$$\tilde{J}(\tilde{x}_0, \tilde{K}) = \tilde{x}_0^T \tilde{H} \tilde{x}_0 \tag{35}$$

where $\tilde{H}$ is the unique, symmetric, and positive definite solution of the Liapunov matrix equation

$$(\tilde{A} - \tilde{B}\tilde{K})^T \tilde{H} + \tilde{H}(\tilde{A} - \tilde{B}\tilde{K}) + \tilde{C}^T\tilde{C} + \tilde{K}^T R \tilde{K} = 0 \tag{36}$$

Multiplying Eq. (36) by U^T from the left and by U from the right, and using the aggregation conditions of Eq. (27), we get

$$(A - BK)^T H + H(A - BK) + C^T C + K^T R K = 0 \tag{37}$$

where

$$K = \bar{K}U, \qquad H = U^T \bar{H} U \tag{38}$$

Comparing Eq. (37) with Eq. (11), and keeping in mind that $Q = C^T C$, we conclude that

$$J(x_0, \bar{K}U) = x_0^T U^T \bar{H} U x_0 = \bar{J}(Ux_0, \bar{K}) \tag{39}$$

In particular, if $\bar{K} = \bar{K}^\circ$, the optimal feedback matrix for the problem $\{\bar{S}, \bar{J}\}$, then Eq. (39) gives

$$J(x_0, \bar{K}^\circ U) = \bar{J}(Ux_0, \bar{K}^\circ) = \bar{J}^\circ(Ux_0) \tag{40}$$

Since $\bar{J}^\circ(Ux_0)$ is the global minimum of $\bar{J}$, Eq. (40) further implies that

$$\bar{K}^\circ U = K^\circ \tag{41}$$

and

$$J(x_0, \bar{K}^\circ U) = J^\circ(x_0) = \bar{J}^\circ(Ux_0) \tag{42}$$

which establish the relation between the optimal controls and costs of the two optimization problems $\{S, J\}$ and $\{\bar{S}, \bar{J}\}$ when $\bar{S}$ is an aggregation of S. This result could have been achieved directly by comparing the Riccati equations corresponding to the two problems. We preferred the preceding derivation, because we can reproduce it in the suboptimality context outlined in the next section.

Similarly, if $\tilde{S}$ is a restriction of S, we can show that

$$J(V\tilde{x}_0, K) = \tilde{J}(\tilde{x}_0, KV) \tag{43}$$

provided that costs are finite. In particular,

$$K^\circ V = \tilde{K}^\circ \tag{44}$$

and

$$J^\circ(V\tilde{x}_0) = \tilde{J}^\circ(x_0) \tag{45}$$

which relates the optimal control and costs for the two optimization problems.

IV. Suboptimality of Reduced-Order Models

One of the difficulties in applying the inclusion principle to model reduction is the fact that it may be overly restrictive and conditions for a given system S to have a contraction $\bar{S}$ do not hold. A natural way to resolve this problem is to introduce an approximate contraction procedure and evaluate the suboptimality of the end result.

It is clear from the aggregation conditions of Eq. (27) that the system S is aggregable to a smaller system $\bar{S}$ if, and only if, it is unobservable. Thus, if the problem $\{S, J\}$ is such that all the state variables are observed in the performance index J, then it cannot be reduced to a smaller problem. In order to produce an approximate aggregation,[8] we split the system matrix A into two matrices as

$$A = A_N + A_S \tag{46}$$

By choosing the "surplus matrix" A_S appropriately, the "new" matrix $A_N = A - A_S$ can be made such that the system

$$S_N\!: \ \dot{x} = A_N x + Bu, \qquad y = Cx \tag{47}$$

is aggregable to a smaller system

$$\bar{S}\!: \ \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \qquad \bar{y} = \bar{C}\bar{x} \tag{48}$$

The aggregation conditions of Eq. (27) require that

$$U(A - A_S) = \bar{A}U, \qquad C = \bar{C}U \tag{49}$$

for some U, $\bar{A}$, and $\bar{C}$. The choice of U is arbitrary provided the second equation in Eq. (49) is satisfied by some $\bar{C}$. An obvious choice is $U = C$, $\bar{C} = I$, which produces $\bar{S}$ of smallest dimension. This may, however, require the elements of A_S to be large compared to those of A, which is not desirable since we want to make $\bar{S}$ "close" to S. Alternatively, making use of the concept of chain aggregation,[9] we can choose $U = [C^T U_C^T]^T$, $\bar{C} = [I \ 0]$, where U_C is such that U has full row rank. In this case, a suitable choice of U_C can make it possible to satisfy Eq. (49) with A_S sufficiently small.

With U fixed, from Eq. (49) we get

$$UA_S = UA - \bar{A}U \tag{50}$$

which suggests that $\bar{A}$ should be chosen as

$$\bar{A} = UAU^T(UU^T)^{-1} \tag{51}$$

in order to minimize $\|UA_S\|$. With $\bar{A}$ as in Eq. (51), Eq. (50) reduces to

$$UA_S = UA[I - U^T(UU^T)^{-1}U] \tag{52}$$

the minimum-norm solution of which is

$$A_S = U^T(UU^T)^{-1}UA[I - U^T(UU^T)^{-1}U] \tag{53}$$

In summary, we showed that by subtracting a matrix A_S from the nominal matrix A, we can produce an approximate aggregate $\bar{S}$ of S from some U and $\bar{C}$. The matrix $\bar{A}$ of $\bar{S}$ can be determined from Eq. (51) and $\bar{B} = UB$. We can then proceed to solve the reduced-order problem $\{\bar{S}, \bar{J}\}$ and produce the low-order gain matrix $\bar{K}^\circ$. Then, the optimization problem $\{S, J\}$ can be solved to serve as a reference for suboptimality of the system S driven by the control $u = -\bar{K}^\circ Ux$ of Eq. (21), that is obtained from the low-order design $\bar{K}^\circ$. Suboptimality conditions are provided by theorem 1 or 2.

An alternative to the approximate aggregation just considered is to aggregate the weighting matrix $C^T C$ of the state of S in the performance index J, as proposed by Aoki.[8] This is done by choosing an aggregation matrix U with full row rank, such that

$$UA = \bar{A}U \tag{54}$$

for some matrix $\bar{A}$, and computing $\bar{B}$ as

$$\bar{B} = UB \tag{55}$$

Conditions of Eqs. (54) and (55) are part of the aggregation conditions in Eq. (27). However, since we assumed that S is not perfectly aggregable to a smaller system, there is no $\bar{C}$ that satisfies the last aggregation condition, namely, $C = \bar{C}U$. Here, we deviate from perfect aggregation, and choose $\bar{C}$ such that $\|C - \bar{C}U\|$ is minimum, i.e.,

$$\bar{C} = CU^T(UU^T)^{-1} \tag{56}$$

The matrices $\{\bar{A}, \bar{B}, \bar{C}\}$ of Eqs. (54-56) constitute an approximate aggregate system $\bar{S}$ of Eq. (49), with which we associate the cost $\bar{J}$ in Eq. (34).

Now, consider the Riccati equation

$$\bar{A}^T\bar{P} + \bar{P}\bar{A} - \bar{P}\bar{B}R^{-1}\bar{B}^T\bar{P} + \bar{C}^T\bar{C} = 0 \tag{57}$$

associated with the aggregate problem $\{\bar{S}, \bar{J}\}$. Multiplying

Eq. (57) by U^T from the left and by U from the right, and using Eqs. (54-56) we obtain

$$A^T(U^T\bar{P}U) + (U^T\bar{P}U)A - (U^T\bar{P}U)BR^{-1}B^T(U^T\bar{P}U) + \Pi C^T C\Pi = 0 \tag{58}$$

where the matrix

$$\Pi = U^T(UU^T)^{-1}U \tag{59}$$

is a projection on $\mathbb{R}^n$. Comparing Eq. (58) with the Riccati equation, Eq. (5), of the original problem $\{\mathcal{S}, J\}$, we observe that $U^T\bar{P}U$ can be considered as an approximate of P provided $C\Pi$ is close to C. Thus,

$$K = \bar{K}^\circ U = R^{-1}\bar{B}^T\bar{P}U = R^{-1}B^T(U^T\bar{P}U) \tag{60}$$

is an approximate to the optimal feedback K° in Eq. (4), the suboptimality of which can be tested using theorem 1 or 2.

We note that condition Eq. (54) requires that U consist of reciprocal eigenvectors of A, and $\bar{A}$ retain those eigenvalues of A that correspond to these eigenvectors. Thus, the aggregation of the performance index as explained earlier consists of choosing an eigenspace of A and modifying the performance index to reflect the contribution of the selected modes to the cost. In this sense, this type of approximate aggregation is closely related to Skelton's model reduction scheme through modal cost analysis of open-loop and closed-loop systems.[10,11] For a given order $\bar{n}$ of the reduced-order model, the $\bar{n}$ modes that contribute most to the cost J can be determined by computing U such that $\|C - C\Pi\|$ is minimum subject to Eq. (54), where Π is given by Eq. (59). However, whether the modes determined by this U matrix will be the most significant ones in the closed-loop system is not clear. One way of deciding on the best approximate aggregation might be to compute the degree of suboptimality μ for each of the $n!/(n-\bar{n})!\bar{n}!$ possible choices of U, and to choose the one with minimum μ. However, noting that the contribution of a mode to the closed-loop cost depends on how controllable as well as how observable it is, a more refined model-reduction procedure can be developed in the suboptimality framework via singular-value decomposition.

The singular-value decomposition of a system using balanced representation has been developed by Moore[12] and later studied in detail by others (see Refs. 13 and 14). In simple terms, a balanced system is one which is as controllable as it is observable, where the measure of controllability and observability is provided by the singular values[12] of the respective Gramians.

To illustrate the idea, suppose $\mathcal{S}$ is stable and let

$$\Sigma_C = \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt \tag{61}$$

$$\Sigma_O = \int_0^\infty e^{A^Tt}C^TCe^{At}dt \tag{62}$$

denote the controllability and observability Gramians of $\mathcal{S}$, respectively. Now consider the product $\Sigma_O\Sigma_C$. Assuming $\mathcal{S}$ is controllable and observable (otherwise, a preliminary procedure can be used to reduce the system order by deleting uncontrollable and/or unobservable part), Σ_O and Σ_C are positive definite matrices, and, therefore, all the eigenvalues of $\Sigma_O\Sigma_C$ are positive. Let these eigenvalues be denoted by σ_i^2 with $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n > 0$. The positive square roots σ_i of the eigenvalues of $\Sigma_O\Sigma_C$ are called the singular values of the system $\mathcal{S}$.

It was shown by Moore[12] that there always exists a coordinate frame in which

$$\Sigma_C = \Sigma_O = \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{63}$$

The representation of $\mathcal{S}$ in this coordinate frame is said to be balanced.

Now, suppose the system $\mathcal{S}$ has a balanced representation, and let

$$\Sigma = \text{diag}(\Sigma_1, \Sigma_2) \tag{64}$$

where

$$\Sigma_1 = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\bar{n}}), \quad \Sigma_2 = \text{diag}(\sigma_{\bar{n}+1}, \sigma_{\bar{n}+2}, \ldots, \sigma_n) \tag{65}$$

Let us consider the corresponding partition of $\{A,B,C\}$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = [C_1 \;\; C_2] \tag{66}$$

If $\sigma_{\bar{n}} \gg \sigma_{\bar{n}+1}$ in Eq. (65), then we can consider $\{A_{11}, B_1, C_1\}$ as the most controllable and observable part of the system $\mathcal{S}$. Thus, the $\bar{n}$th order system $\bar{\mathcal{S}} = \{\bar{A}, \bar{B}, \bar{C}\}$ with $\bar{A} = A_{11}$, $\bar{B} = B_1$, $\bar{C} = C_1$, can be taken as a good reduced-order model for $\mathcal{S}$. In fact, it is easy to see from Eq. (64) that $\bar{\mathcal{S}}$ is a good approximation to $\mathcal{S}$ in the sense that the differences between the controllability and observability Gramians of $\bar{\mathcal{S}}$ and $\mathcal{S}$ are minimum in the norm square sense.

Once a reduced-order model $\bar{\mathcal{S}}$ of the system $\mathcal{S}$ is obtained through the singular-value decomposition of a balanced representation of $\mathcal{S}$, a suboptimal control can be designed for $\mathcal{S}$ following the procedure of Sec. II. This approach to model reduction is particularly promising since it allows for generating systematically a sequence of reduced-order models consisting of smaller and smaller portions of the A, B, C matrices in Eq. (66). Computing the degree of suboptimality at each step one can compromise between the degree of suboptimality and the order of the reduced model.

Finally, we note that order reduction through singular-value decomposition can be considered as a combination of the previous two approaches, where we first subtract from the A matrix in Eq. (66) the surplus matrix

$$A_S = \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} \tag{67}$$

so that with $U = [I_{\bar{n}} \; 0]$, $U(A - A_S) = \bar{A}U$, and then choose $\bar{C}$ as in Eq. (56), which yields $\bar{C} = C_1$. However, this procedure is not justified unless $\mathcal{S}$ is balanced.

V. Application

To illustrate the validation of model-reduction schemes using the concept of suboptimality, we consider the feedback design of a large space structure described in Ref. 15. The mathematical model is represented as a set of nine second-order equations

$$\ddot{z}_i + d_i\dot{z}_i + \omega_i^2 z_i = b_i^T u, \qquad i = 1,2,\ldots,9 \tag{68}$$

where $z_i(t)$ are the vibration modes with damping d_i and frequency ω_i, and $u(t)$ is the two-dimensional control input such that b_i^Tu is the scalar control torque on the ith mode. The parameters d_i, ω_i, and b_i^T are tabulated in Table 1.

Letting $x_i = [z_i \dot{z}_i]^T$, and $x = [x_1^T, x_2^T, \ldots, x_9^T]^T$, Eq. (68) can be written compactly as

$$\dot{x} = Ax + Bu \tag{69}$$

where $A = \text{diag}\{A_1, A_2, \ldots, A_9\}$ with

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -d_i \end{bmatrix} \tag{70}$$

and $B=[B_1^T, B_2^T, ..., B_9^T]^T$ with

$$B_i = \begin{bmatrix} 0 \\ b_i^T \end{bmatrix} \quad (71)$$

We choose the output matrix to be of the form $C = \text{diag}\{c_1^T, c_2^T, ..., c_9^T\}$, with

$$C_i^T = [c_{i1} \; 0] \quad (72)$$

to reflect the fact that we penalize only z_i's in the performance index. Equation (69), together with the output equation

$$y = Cx \quad (73)$$

describes the system $\mathcal{S}$ in Eq. (22).

We now apply to $\mathcal{S}$ the model-reduction scheme based on the balanced representation of $\mathcal{S}$. However, before doing this we note from Table 1 that the first three modes are undamped, i.e., the blocks A_1, A_2, and A_3 in Eq. (70) are not stable. This prevents the model-reduction scheme from being applied directly, for it requires the system to be stable. To avoid this difficulty, we partition the state vector as $x=[x_u^T, x_d^T]^T$, where $x_u=[x_1^T x_2^T x_3^T]^T$ and $x_d=[x_4^T x_5^T ... x_9^T]^T$ correspond, respectively, to the undamped and damped modes. Partitioning A and B of Eq. (69) and C of Eq. (73) accordingly, we represent $\mathcal{S}$ as the parallel connection of two systems $\mathcal{S}_u$ and $\mathcal{S}_d$, which are described by

$$\mathcal{S}_u: \; \dot{x}_u = A_u x_u + B_u u, \qquad y_u = C_u x_u \quad (74)$$

$$\mathcal{S}_d: \; \dot{x}_d = A_d x_d + B_d u, \qquad y_d = C_d x_d \quad (75)$$

Now since $\mathcal{S}_d$ is stable, we can obtain a reduced order model $\bar{\mathcal{S}}_d$ for $\mathcal{S}_d$ based on its balanced representation. The parallel connection of $\mathcal{S}_u$ and $\bar{\mathcal{S}}_d$ then gives a reduced-order model $\bar{\mathcal{S}}$ of $\mathcal{S}$. In other words, we keep the undamped modes in the reduced-order model and add to these a part of the damped modes.

We choose $R = 10^{-5} I$, and c_{i1} in Eq. (72) as

$$\begin{aligned} c_{i1} &= 1, & i &= 1,2,3 \\ c_{i1} &= 0.1, & i &= 4,5,...,9 \end{aligned} \quad (76)$$

to penalize the undamped modes more than the damped ones. The optimal gain matrix and the eigenvalues of the optimal system are shown in Table 2, along with the singular values of the product $\Sigma_O \Sigma_C$ of the observability and controllability Gramians of $\mathcal{S}_d$. From these values we observe that the last four singular values are very small compared to the first eight, which suggests that an eighth-order reduced model should adequately represent $\mathcal{S}_d$. The results of suboptimal design based on an eighth-order reduced model of $\mathcal{S}_d$ are shown in Table 3. The suboptimal gains and eigenvalues of the suboptimal system coincide with those of the optimal ones up to the fourth decimal place, producing a degree of suboptimality $\mu = 1.000$ and thus confirming our observation. The results of the suboptimal design based on a second-order

Table 1 Parameters of the model of the large space structure

i	d_i, $\times 10^{-2}$	ω_i	b_i^T, $\times 10^{-4}$	
1	0	0	3.460	3.460
2	0	0.148	−9.120	34.400
3	0	0.288	−1.300	0.548
4	1.414	1.414	−1.400	36.000
5	1.829	1.829	−4.070	−34.700
6	2.846	2.846	28.500	39.500
7	3.831	3.831	71.800	−24.600
8	8.798	8.798	-8.000×10^{-4}	-1.150×10^{-5}
9	9.613	9.613	−0.172	-4.130×10^{-3}

Table 2 Optimal design and singular values of $\Sigma_O \Sigma_C$

		Optimal gain matrix (columns 1-6)			
0.2998*E* 03	0.1398*E* 04	−0.9208*E* 02	−0.2136*E* 03	−0.2960*E* 03	0.3121*E* 03
0.1005*E* 03	0.3982*E* 03	0.2953*E* 03	0.3899*E* 03	−0.6539*E* 02	0.2092*E* 02
		Optimal gain matrix (columns 6-12)			
0.3381*E* 01	0.2121*E* 01	0.1080*E* 01	−0.2634*E* 01	0.3069*E* 01	0.3647*E* 01
0.2261*E* 02	0.8127*E* 01	−0.1787*E* 02	−0.7468*E* 01	−0.9151*E* 01	−0.4424*E* 01
		Optimal gain matrix (columns 12-18)			
0.4580*E* 01	0.5322*E* 01	−0.4572*E*−05	−0.5848*E*−05	−0.7579*E*−03	−0.9656*E*−03
0.4559*E* 01	−0.1416*E* 01	0.1311*E*−05	−0.1068*E*−06	0.2002*E*−03	−0.2618*E*−04
		Optimal eigenvalues			
		Re	Im		
		−0.4807*E*−01	0.9613*E* 01		
		−0.4807*E*−01	−0.9613*E* 01		
		−0.4399*E*−01	0.8798*E* 01		
		−0.4399*E*−01	−0.8798*E* 01		
		−0.4132*E*−01	0.3268*E* 01		
		−0.4132*E*−01	−0.3268*E* 01		
		−0.3030*E*−01	0.2846*E* 01		
		−0.3030*E*−01	−0.2846*E* 01		
		−0.3014*E*−01	0.1829*E* 01		
		−0.3014*E*−01	−0.1829*E* 01		
		−0.3587*E*−01	0.1414*E* 01		
		−0.3587*E*−01	−0.1414*E* 01		
		−0.3215*E*−01	0.2843*E* 00		
		−0.3215*E*−01	−0.2843*E* 00		
		−0.7430*E* 00	0.7585*E* 00		
		−0.7430*E* 00	−0.7585*E* 00		
		−0.2568*E* 00	0.2629*E* 00		
		−0.2568*E* 00	−0.2629*E* 00		
		Singular values of $\Sigma_O \Sigma_C$			
0.1364*E*−03	0.1127*E*−03	0.4021*E*−04	0.1817*E*−04	0.9167*E*−05	0.8128*E*−05
0.7243*E*−05	0.5585*E*−05	0.8753*E*−12	0.8578*E*−12	0.2664*E*−16	0.2610*E*−16

Table 3 Suboptimal design I

Order of the reduced system = 8

Suboptimal gain matrix (columns 1-6)

0.2998E 03	0.1398E 04	−0.9208E 02	−0.2136E 03	−0.2960E 03	0.3121E 03
0.1005E 03	0.3892E 02	0.2953E 03	0.3899E 03	−0.6539E 02	0.2092E 02

Suboptimal gain matrix (columns 6-12)

0.3381E 01	0.2121E 01	0.1080E 01	−0.2634E 01	0.3069E 01	0.3647E 01
0.2261E 02	0.8127E 01	−0.1787E 02	−0.7468E 01	−0.9151E 01	−0.4424E 01

Suboptimal gain matrix (columns 12-18)

0.4580E 01	0.5322E 01	0.1637E−03	0.2118E−04	0.7986E−06	−0.4600E−07
0.4559E 01	−0.1416E 01	−0.1162E−04	−0.4589E−04	0.3365E−06	0.1004E−06

Suboptimal eigenvalues

Re	Im
−0.4807E−01	0.9613E 01
−0.4807E−01	−0.9613E 01
−0.4399E−01	0.8798E 01
−0.4399E−01	−0.8798E 01
−0.4132E−01	0.3268E 01
−0.4132E−01	−0.3268E 01
−0.3030E−01	0.2846E 01
−0.3030E−01	−0.2846E 01
−0.3014E−01	0.1829E 01
−0.3014E−01	−0.1829E 01
−0.3587E−01	0.1414E 01
−0.3587E−01	−0.1414E 01
−0.3215E−01	0.2843E 00
−0.3215E−01	−0.2843E 00
−0.7430E 00	0.7585E 00
−0.7430E 00	−0.7585E 00
−0.2568E 00	0.2629E 00
−0.2568E 00	−0.2629E 00

Degree of suboptimality = 0.1000E 01

Table 4 Suboptimal design II

Order of the reduced system = 2

Suboptimal gain matrix (columns 1-6)

0.3005E 03	0.1400E 04	−0.0903E 02	−0.2090E 03	−0.2964E 03	0.3133E 03
0.9836E 02	0.3764E 03	0.2963E 03	0.3783E 03	−0.6326E 02	0.2004E 02

Suboptimal gain matrix (columns 6-12)

0.7902E−01	0.9640E−01	−0.7305E−02	0.3679E−01	−0.5248E−02	−0.3086E−01
0.4662E−01	−0.8469E−02	0.1237E−02	0.2041E−01	−0.2614E−02	−0.1764E−01

Suboptimal gain matrix (columns 12-18)

0.1034E 01	0.4096E−01	0.3470E−04	−0.6722E−06	0.1853E−06	−0.1055E−07
0.6037E 00	0.5238E−02	0.2045E−04	−0.4006E−06	0.1093E−06	−0.6106E−08

Suboptimal eigenvalues

Re	Im
−0.4807E−01	0.9613E 01
−0.4807E−01	−0.9613E 01
−0.4399E−01	0.8798E 01
−0.4399E−01	−0.8798E 01
−0.1940E−01	0.3269E 01
−0.1940E−01	−0.3269E 01
−0.1422E−01	0.2846E 01
−0.1422E−01	−0.2846E 01
−0.9109E−02	0.1829E 01
−0.9109E−02	−0.1829E 01
−0.7048E−02	0.1414E 01
−0.7048E−02	−0.1414E 01
−0.7441E 00	0.7583E 00
−0.7441E 00	−0.7583E 00
−0.2568E 00	0.2628E 00
−0.2568E 00	−0.2628E 00
−0.3250E−01	0.2843E 00
−0.3250E−01	−0.2843E 00

Degree of suboptimality = 0.3317E 00

reduced model of $\mathcal{S}_d$ are shown in Table 4, which shows a 67% loss of performance quality corresponding to an 83% reduction in the order of $\mathcal{S}_d$.

VI. Conclusions

Almost exclusively, the available model-reduction schemes are concerned with the open-loop behavior of systems and, therefore, lack the justification of producing lower-order models which would also represent adequately the closed-loop performance of the systems they approximate. The suboptimality concept outlined in this paper provides a means for validation of reduced-order models by comparing the performance of closed-loop reduced model with that of the full-order system. The advantages of this validation approach are as follows.

1) It is particularly suitable for measuring effectiveness of controls under information structure constraints.

2) It can be successfully applied to interconnected systems to produce suboptimal control.

3) It provides robustness measures such as gain and phase margin, explicitly in terms of the degree of suboptimality.

A drawback of the suboptimality approach is the necessity to solve the control problem for the full-order system to be used as reference. This may not be very important unless the order of the system is extremely large or it contains uncertain parameters, but becomes a challenging difficulty in these cases.

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